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REVIEW

Approximation of common solutions for system of equilibrium problems and fixed-point problems

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Abstract In this paper, we complement the result of Zhu and Chang (J Ineq Appl 2013:146, 2013) by proving strong convergence theorems for approximation of a fixed point of a left Bregman strongly relatively nonexpansive mapping which is also a solution to a finite system of equilibrium problems in the framework of reflexive real Banach spaces using the Halpern–Mann's iterations used in Zhu and Chang (J Ineq Appl 2013:146, 2013). We also discuss the approximation of a common fixed point of a family of left Bregman strongly nonexpansive mappings which is also solution to a finite system of equilibrium problems in reflexive real Banach spaces. Our results complement many known recent results in the literature.

Keywords Left Bregman strongly relatively nonexpansive mapping · Left Bregman projection · Equilibrium problem · Banach spaces

Mathematics Subject Classification (2000) 47H06 · 47H09 · 47J05 · 47J25

Introduction

In this paper, let C be a nonempty, closed and convex subset of a real reflexive Banach space E with the dual space E^*. The norm and the dual pair between E and E^* are denoted by $\|.\|$ and $\langle ., . \rangle$, respectively. Let $T : C \to C$ be a nonlinear mapping. Denote by $F(T) := \{x \in C : Tx = x\}$ the set of fixed points of T. A mapping T is said to be *nonexpansive* if $||Tx - Ty|| \leq ||x - y||, \quad \forall x, y \in C$.

In 1994, Blum and Oettli [8] firstly studied the equilibrium problem: finding $x \in C$ such that

$$g(x, y) \geq 0, \quad \forall y \in C, \tag{1}$$

where $g : C \times C \to \mathbb{R}$ is a functional. Denote the set of solutions of the problem (1) by $EP(g)$. Since then, various equilibrium problems have been investigated. It is well known that equilibrium problems and their generalizations have been important tools for solving problems arising in the fields of linear or nonlinear programming, variational inequalities, complementary problems, optimization problems, fixed-point problems and have been widely applied to physics, structural analysis, management science and economics etc (see, for example [8, 26, 27]). One of the most important and interesting topics in the theory of equilibria is to develop efficient and implementable algorithms for solving equilibrium problems and their generalizations (see, e.g., [8, 26, 27, 47] and the references therein). Since the equilibrium problems have very close connections with both the fixed-point problems and the variational inequalities problems, finding the common elements of these problems has drawn many people's attention and has become one of the hot topics in the related fields in the past few years (see, e.g., [7, 17, 21, 29, 30, 31, 39, 42, 43, 44, 48] and the references therein).

In 1967, Bregman [11] discovered an elegant and effective technique for using of the so-called Bregman distance function D_f (see "Preliminaries", Definition 2.1) in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique has been applied in various ways to design and analyze not only iterative algorithms for solving feasibility and optimization

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problems, but also algorithms for solving variational inequalities, for approximating equilibia, for computing fixed points of nonlinear mappings and so on (see, e.g., [3, 17, 41, 42, 43] and the references therein). In 2005, Butnariu and Resmerita [12] presented Bregman-type iterative algorithms and studied the convergence of the Bregman-type iterative method of solving some nonlinear operator equations.

Recently, using the Bregman projection, Reich and Sabach [35] presented the following algorithms for finding common zeroes of maximal monotone operators $A_i : E \to 2^{E^*}, (i = 1, 2, \ldots, N)$ in a reflexive Banach space E, respectively:

$$\begin{cases} x_0 \in E, \\ y_n^i = Res_{\lambda_n^i}^f (x_n + e_n^i), \\ C_n^i = \{z \in E : D_f(z, y_n^i) \le D_f(z, x_n + e_n^i)\}, \\ C_n = \cap_{i=1}^N C_n^i, \\ Q_n = \{z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \le 0\}, \\ x_{n+1} = proj_{C_n \cap Q_n}^f x_0, \quad n \ge 0 \end{cases} \tag{2}$$

and

$$\begin{cases} x_0 \in E, \\ \eta_n^i = \xi_n^i + \dfrac{1}{\lambda_n^i} (\nabla f(y_n^i) - \nabla f(x_n)), \quad \xi_n^i \in A_i y_n^i, \\ \omega_n^i = \nabla f^*(\lambda_n^i \eta_n^i + \nabla f(x_n)), \\ C_n^i = \{z \in E : D_f(z, y_n^i) \le D_f(z, x_n + e_n^i)\}, \\ C_n = \cap_{i=1}^N C_n^i, \\ Q_n = \{z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \le 0\}, \\ x_{n+1} = proj_{C_n \cap Q_n}^f x_0, \quad n \ge 0 \end{cases} \tag{3}$$

where $\{\lambda_n^i\}_{i=1}^N \subset (0, +\infty), \{e_n^i\}_{i=1}^N$ is an error sequence in E with $e_n^i \to 0$ and, $proj_C^f$ is the Bregman projection with respect to f from E onto a closed and convex subset C. Further, under some suitable conditions, they obtained two strong convergence theorems of maximal monotone operators in a reflexive Banach space. Reich and Sabach [36] also studied the convergence of two iterative algorithms for finitely many Bregman strongly nonexpansive operators in a Banach space. In [37], Reich and Sabach proposed the following algorithms for finding common fixed points of finitely many Bregman firmly nonexpansive operators $T_i : C \to C \quad (i = 1, 2, \ldots, N)$ in a reflexive Banach space E if $\cap_{i=1}^N F(T_i) \neq \emptyset$:

$$\begin{cases} x_0 \in E, \\ Q_0^i = E, \quad i = 1, 2, \ldots, N, \\ y_n^i = T_i(x_n + e_n^i), \\ Q_{n+1}^i = \{z \in Q_n^i : \langle \nabla f(x_n + e_n^i) - \nabla f(y_n^i), z - y_n^i \rangle \le 0\}, \\ C_n = \cap_{i=1}^N C_n^i, \\ x_{n+1} = proj_{C_{n+1}}^f x_0, \quad n \ge 0. \end{cases} \tag{4}$$

Under some suitable conditions, they proved that the sequence $\{x_n\}$ generated by (4) converges strongly to $\cap_{i=1}^N F(T_i)$ and applied the result to the solution of convex feasibility and equilibrium problems.

In 2011, Chen et al. [18] introduced the concept of weak Bregman relatively nonexpansive mappings in a reflexive Banach space and gave an example to illustrate the existence of a weak Bregman relatively nonexpansive mapping and the difference between a weak Bregman relatively nonexpansive mapping and a Bregman relatively nonexpansive mapping. They also proved the strong convergence of the sequences generated by the constructed algorithms with errors for finding a fixed point of weak Bregman relatively nonexpansive mappings and Bregman relatively nonexpansive mappings under some suitable conditions.

Recently, Suantai et al. [40] considered strong convergence results for Bregman strongly nonexpansive mappings in reflexive Banach spaces by Halpern's iteration.In particular, they proved the following theorem.

Theorem 1.1 *Let E be a real reflexive Banach space and $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let T be a Bregman strongly nonexpansive mapping on E such that $F(T) = \widehat{F}(T) \neq \emptyset$ Suppose that $u \in E$ and define the sequence $\{x_n\}$ as follows: $x_1 \in E$ and*

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(Tx_n)), \quad n \ge 1, \tag{5}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $P_C^f(u)$, where $P_{F(T)}^f$ is the Bregman projection of E onto $F(T)$.

Furthermore, using the Theorem 1.1, Suantai et al. [40] obtained some convergence theorems for a family of Bregman strongly nonexpansive mappings and gave some applications concerning the problems of finding zeroes of maximal monotone operators and equilibrium problems.

Very recently, Zhu and Chang [49] considered strong convergence results for Bregman strongly nonexpansive mappings in reflexive Banach spaces by modifying Halpern and Mann's iterations. Furthermore, they gave some applications concerning the problems of finding zeros of maximal monotone operators and equilibrium problems. In particular, they proved the following theorem.

Theorem 1.2 *Let E be a real reflexive Banach space and $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let T be a Bregman strongly nonexpansive mapping on E such that $F(T) = \widehat{F}(T) \neq \emptyset$. Suppose that $u \in E$ and define the sequence $\{x_n\}$ as follows: $x_1 \in E$ and*

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)(\beta_n \nabla f(x_n) + (1 - \beta_n)\nabla f(Tx_n))), \quad n \geq 1, \tag{6}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ are sequences in (0,1) *satisfying*

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$;

$(C2)$ $\sum_{n=1}^{\infty} \alpha_n = \infty$;

$(C3)$ $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$

Then $\{x_n\}$ *converges strongly to* $P^f_{F(T)}(u)$, *where* $P^f_{F(T)}$ *is the Bregman projection of E onto F(T).*

Motivated by the results of Suantai et al. [40] and Zhu and Chang [49], the purpose of this paper is to prove strong convergence theorems for approximation of a fixed point of a left Bregman strongly relatively nonexpansive mapping which is also a solution to a finite system of equilibrium problems in the framework of reflexive real Banach spaces. We also discuss the approximation of a common fixed point of a family of left Bregman strongly nonexpansive mappings which is also solution to a finite system of equilibrium problems in reflexive real Banach spaces. Our results complement many known recent results in the literature.

Preliminaries

In this section, we present the basic notions and facts that are needed in the sequel. The norms of E and E^*, its dual space, are denoted by $\|.\|$ and $\|.\|_*$, respectively. The pairing $\langle \xi, x\rangle$ is defined by the action of $\xi \in E^*$ at $x \in E$, that is, $\langle \xi, x\rangle := \xi(x)$. The *domain* of a convex function $f : E \to \mathbb{R}$ is defined to be

$$\text{dom}\, f := \{x \in E : f(x) < +\infty\}.$$

When dom $f \neq \emptyset$, we say that f is proper. The *Fenchel conjugate* function of f is the convex function $f^* : E \to \mathbb{R}$ defined by

$$f^*(\xi) = sup\{\langle \xi, x\rangle - f(x) : x \in E\}.$$

It is not difficult to check that when f is proper and lower semicontinuous, so is f^*. The function f is said to be *cofinite* if dom $f^* = E^*$.

Let $x \in \text{int dom} f$, that is, let x belong to the interior of the domain of the convex function $f : E \to (-\infty, +\infty]$. For any $y \in E$, we define the *directional derivative* of f at x by

$$f^o(x, y) := \lim_{t\to 0^+} \frac{f(x+ty) - f(x)}{t}. \tag{7}$$

If the limit as $t \to 0^+$ in (7) exists for each y, then the function f is said to be *Gâteaux differentiable* at x. In this case, the gradient of f at x is the linear function $\nabla f\,(x)$, which is defined by $\langle \nabla f(x), y\rangle := f^o(x, y)$ for all $y \in E$ [19, Definition 1.3, page 3]. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in \text{int dom} f$. When the limit as $t \to 0$ in (7) is attained uniformly for any $y \in E$ with $\|y\| = 1$, we say that f is *Fréchet differentiable* at x. Throughout this paper, $f : E \to (-\infty, +\infty]$ is always an *admissible function*, that is, a proper, lower semicontinuous, convex and Gâteaux differentiable function. Under these conditions, we know that f is continuous in int dom f (see [3], Fact 2.3, page 619).

The function f is said to be *Legendre* if it satisfies the following two conditions.

- $(L1)$ int dom $f \neq \emptyset$ and the subdifferential ∂f are single-valued on its domain.
- $(L2)$ int dom $f^* \neq \emptyset$ and ∂f^* are single-valued on its domain.

The class of Legendre functions in infinite-dimensional Banach spaces was first introduced and studied by Bauschke, Borwein and Combettes in [3]. Their definition is equivalent to conditions $(L1)$ and $(L2)$ because the space E is assumed to be reflexive (see [3], Theorems 5.4 and 5.6, page 634). It is well known that in reflexive spaces $\nabla f = (\nabla f^*)^{-1}$ (see [9], page 83). When this fact is combined with conditions $(L1)$ and $(L2)$, we obtain

$$\text{ran}\nabla f = \text{dom}\,\nabla f^* = \text{int dom} f^* \text{ and ran}\,\nabla f^* = \text{dom}\,\nabla f = \text{int dom} f.$$

It also follows that f is Legendre if and only if f^* is Legendre (see [3], Corollary 5.5, page 634) and that the functions f and f^* are Gateaux differentiable and strictly convex in the interior of their respective domains. When the Banach space E is smooth and strictly convex, in particular, a Hilbert space, the function $(\frac{1}{p})||.||^p$ with $p \in (1, \infty)$ is Legendre (cf. [3], Lemma 6.2, page 639). For examples and more information regarding Legendre functions, see, for instance, [3, 4].

Definition 2.1 The bifunction $D_f : \text{dom}\, f \times \text{int dom}\, f \to [0, +\infty)$, which is defined by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle, \tag{8}$$

is called the *Bregman distance* (cf. [11, 15]).

The Bregman distance does not satisfy the well-known properties of a metric, but it does have the following important property, which is called the three point identity: for any $x \in \text{dom} f$ and $y, z \in \text{int dom} f$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle. \tag{9}$$

According to [13], Section 1.2, page 17 (see also [14]), the *modulus of total convexity* of f is the bifunction υ_f : $\operatorname{int} \operatorname{dom} f \times [0, +\infty) \to [0, +\infty]$ which is defined by
$\upsilon_f(x, t) := \inf\{D_f(y, x) : y \in \operatorname{dom} f, ||y - x|| = t\}$.

The function f is said to be *totally convex at a point* $x \in \operatorname{int} \operatorname{dom} f$ if $\upsilon_f(x, t) > 0$ whenever $t > 0$. The function f is said to be *totally convex* when it is totally convex at every point $x \in \operatorname{int} \operatorname{dom} f$. This property is less stringent than uniform convexity (see [13], Section 2.3, page 92).

Examples of totally convex functions can be found, for instance, in [10, 12, 13]. We remark in passing that f is totally convex on bounded subsets if and only if f is uniformly convex on bounded subsets (see [12], Theorem 2.10, page 9).

The *Bregman projection* (cf. [11]) with respect to f of $x \in \operatorname{int} \operatorname{dom} f$ onto a nonempty, closed and convex set $C \subset \operatorname{int} \operatorname{dom} f$ is defined as the necessarily unique vector $proj_C^f(x) \in C$, which satisfies

$$D_f(proj_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}. \tag{10}$$

Similarly to the metric projection in Hilbert spaces, the Bregman projection with respect to totally convex and Gâteaux differentiable functions has a variational characterization (cf. [12], Corollary 4.4, page 23).

Proposition 2.2 (Characterization of Bregman Projections). *Suppose that* $f : E \to (-\infty, +\infty]$ *is totally convex and Gâteaux differentiable in int dom f. Let* $x \in \operatorname{int} \operatorname{dom} f$ *and let* $C \subset \operatorname{int} \operatorname{dom} f$ *be a nonempty, closed and convex set. If* $\hat{x} \in C$, *then the following conditions are equivalent.*

1. *The vector* $\hat{x}$ *is the Bregman projection of* x *onto* C *with respect to f.*
2. *The vector* $\hat{x}$ *is the unique solution of the variational inequality*

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \geq 0 \quad \forall y \in C.$$

3. *The vector* $\hat{x}$ *is the unique solution of the inequality* $D_f(y, z) + D_f(z, x) \leq D_f(y, x) \forall y \in C$.

Recall that the function f is said to be *sequentially consistent* [5] if, for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first is bounded,

$$\lim_{n\to\infty} D_f(x_n, y_n) = 0 \Leftrightarrow \lim_{n\to\infty} ||x_n - y_n|| = 0. \tag{11}$$

Let C be a nonempty, closed and convex subset of E and $g : C \times C \to \mathbb{R}$ a bifunction that satisfies the following conditions:

A1. $g(x,x) = 0$ for all $x \in C$;
A2. g is monotone, i.e., $g(x, y) + g(y, x) \leq 0$ for all $x, y, \in C$;
A3. for each $x, y \in C$, $\lim_{t\to 0} g(tz + (1-t)x, y) \leq g(x, y)$;
A4. for each $x \in C$, $y \mapsto g(x, y)$ is convex and lower semicontinuous.

The resolvent of a bifunction $g : C \times C \to \mathbb{R}$ [19] is the operator $Res_g^f : E \to 2^C$ denoted by

$$Res_g^f(x) = \{z \in C : g(z, y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0 \quad \forall y \in C\}. \tag{12}$$

For any $x \in E$, there exists $z \in C$ such that $z = Res_C^f(x)$; see [36].

Let C be a convex subset of int dom$f$ and let T be a self-mapping of C. A point $p \in C$ is said to be an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}_{n=0}^\infty$ which converges weakly to p and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$.

Recalling that the Bregman distance is not symmetric, we define the following operators.

Definition 2.3 A mapping T with a nonempty asymptotic fixed point set is said to be:

1. *left Bregman strongly nonexpansive* (see [5, 6]) with respect to a nonempty $\widehat{F}(T)$ if

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C, \quad p \in \widehat{F}(T)$$

and if whenever $\{x_n\} \subset C$ is bounded, $p \in \widehat{F}(T)$ and

$$\lim_{n\to\infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0,$$

it follows that

$$\lim_{n\to\infty} D_f(Tx_n, x_n) = 0.$$

According to Martin-Marquez et al. [23], a left Bregman strongly nonexpansive mapping T with respect to a nonempty $\widehat{F}(T)$ is called *strictly left Bregman strongly nonexpansive mapping*.

2. An operator $T : C \to \operatorname{int} \operatorname{dom} f$ is said to be: *left Bregman firmly nonexpansive* (L-BFNE) if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle$$

for any $x, y \in C$, or equivalently,

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x).$$

See [5, 10, 33] for more information and examples of L-BFNE operators (operators in this class are also called D_f-firm and BFNE). For two recent studies of the existence and approximation of fixed points of left Bregman firmly nonexpansive operators, see [24, 33]. It is also known that if T is left Bregman firmly nonexpansive and f is Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E, then $F(T) = \widehat{F}(T)$ and $F(T)$ is closed and convex (see [33]). It also follows that every left Bregman firmly nonexpansive mapping is left Bregman strongly nonexpansive with respect to $F(T) = \widehat{F}(T)$.

Martin-Marquez et al. [23] called the Bregman projection defined in (10) and chracterized by Proposition 2.2 above as the *left Bregman projection* and they denoted the left Bregman projection by $\overleftarrow{Proj}^f_C$.

Let $f : E \to \mathbb{R}$ be a convex, Legendre and Gâteaux differentiable function. Following [1] and [15], we make use of the function $V_f : E \times E^* \to [0, +\infty)$ associated with f, which is defined by

$$V_f(x, x^*) = f(x) - \langle x^*, x\rangle + f^*(x^*), \forall x \in E, x^* \in E^*.$$

Then V_f is nonnegative and $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$ for all $x \in E$ and $x^* \in E^*$. Moreover, by the subdifferential inequality,

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \le V_f(x, x^* + y^*) \tag{13}$$

for all $x \in E$ and $x^*, y^* \in E^*$ (see also [20], Lemmas 3.2 and 3.3). In addition, if $f : E \to (-\infty, +\infty]$ is a proper lower semi-continuous function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semi-continuous and convex function (see [28]). Hence V_f is convex in the second variable. Thus, for all $z \in E$,

$$D_f\Big(z, \nabla f^*\Big(\sum_{i=1}^N t_i \nabla f(x_i)\Big)\Big) \le \sum_{i=1}^N t_i D_f(z, x_i), \tag{14}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$.

Finally, we state some lemmas that will be used in the proof of main results in next section.

Lemma 2.4 (Reich and Sabach [34]) *If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E*.*

Lemma 2.5 (Butnariu and Iusem [13]) *The function f is totally convex on bounded sets if and only if it is sequentially consistent.*

Lemma 2.6 (Reich and Sabach [35]) *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}_{n=1}^\infty$ is bounded, then the sequence $\{x_n\}_{n=1}^\infty$ is also bounded.*

Lemma 2.7 (Reich and Sabach [36]) *Let $f : E \to (-\infty, +\infty)$ be a coercive Legendre function. Let C be a closed and convex subset of E. If the bifunction $g : C \times C \to \mathbb{R}$ satisfies conditions A1–A4, then*

1. *Res^f_C is single-valued;*
2. *Res^f_g is a Bregman firmly nonexpansive mapping;*
3. *$F(Res^f_g) = EP(g)$;*
4. *$EP(g)$ is a closed and convex subset of C;*
5. *for all $x \in E$ and $q \in F(Res^f_g)$,*

$$D_f(q, Res^f_g(x)) + D_f(Res^f_g(x), x) \le D_f(q, x).$$

Lemma 2.8 (Xu [45]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, n \ge 0,$$

where, (1) $\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$; (2) $\limsup \sigma_n \le 0$; (3) $\gamma_n \ge 0$; $(n \ge 0)$, $\sum\gamma_n < \infty$. *Then, $a_n \to 0$ as $n \to \infty$.*

Lemma 2.9 (Mainge [22]) *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{ni} < a_{ni} + 1$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \le a_{m_k+1} \quad \text{and} \quad a_k \le a_{m_k+1}.$$

In fact, $m_k = max\ \{j \le k : a_j < a_{j+1}\}$.

Lemma 2.10 (Suantai et al. [40]) *Let E be a reflexive real Banach space. Let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Suppose T is a left Bregman strongly nonexpansive mappings of C into E such that $F(T) = \widehat{F}(T) \neq \emptyset$. If $\{x_n\}_{n=0}^\infty$ is a bounded sequence such that $x_n - Tx_n \to 0$ and $z := P^f_\Omega(u)$, then*

$$\limsup_{n\to\infty}\langle x_n - z, \nabla f(u) - \nabla f(z)\rangle \le 0.$$

Main results

We first prove the following lemma.

Lemma 3.1 *Let E be a reflexive real Banach space and C a nonempty, closed and convex subset of E. For each $k = 1, 2, ..., N$, let g_k be a bifunction from $C \times C$ satisfying $(A1) - (A4)$. Let $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let T be a left Bregman strongly nonexpansive mapping of C into E such that $F(T) = \widehat{F}(T)$ and $\Omega := F(T) \cap (\cap_{k=1}^N EP(g_k)) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in (0,1). Suppose $\{x_n\}_{n=1}^\infty$ is iteratively generated by u, $u_1 \in E$,*

$$\begin{cases} x_n = Res^f_{g_N} Res^f_{g_{N-1}} \ldots Res^f_{g_2} Res^f_{g_1} u_n, \\ u_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)(\beta_n \nabla f(x_n) + (1 - \beta_n)\nabla f(Tx_n))), \quad n \ge 1. \end{cases} \tag{15}$$

Then, $\{x_n\}_{n=0}^{\infty}$ *is bounded.*

Proof Let $x^* \in \Omega$. By taking $\theta_k^f = Res_{g_k}^f Res_{g_{k-1}}^f \ldots Res_{g_2}^f Res_{g_1}^f$, $k = 1, 2, \ldots, N$ and $\theta_0^f = I$, we obtain $x_n = \theta_N^f u_n$. Using the fact that Res_C^f, $k = 1, 2, \ldots, N$ is a strictly left quasi-Bregman nonexpansive mapping, we obtain from (15) that

$$\begin{aligned} D_f(x^*, x_{n+1}) =& D_f(x^*, \theta_N^f u_{n+1}) \le D_f(x^*, u_{n+1}) \\ =& D_f(x^*, \nabla f^*(\alpha_n \nabla f(u) + \beta_n(1-\alpha_n)\nabla f(x_n) \\ & + (1-\alpha_n)(1-\beta_n)\nabla f(Tx_n)) \\ \le& \alpha_n D_f(x^*, u) + \beta_n(1-\alpha_n) D_f(x^*, x_n) \\ & + (1-\alpha_n)(1-\beta_n) D_f(x^*, Tx_n) \\ \le& \alpha_n D_f(x^*, u) + \beta_n(1-\alpha_n) D_f(x^*, x_n) \\ & + (1-\alpha_n)(1-\beta_n) D_f(x^*, x_n) \\ =& \alpha_n D_f(x^*, u) + (1-\alpha_n) D_f(x^*, x_n) \\ \le& \max\{D_f(x^*, u), D_f(x^*, x_n)\} \\ & \vdots \\ \le& \max\{D_f(x^*, u), D_f(x^*, x_1)\}. \end{aligned} \tag{16}$$

Hence, $\{D_f(x^*, x_n)\}_{n=1}^{\infty}$ is bounded. We next show that the sequence $\{x_n\}$ is also bounded. Since $\{D_f(x^*, x_n)\}_{n=1}^{\infty}$ is bounded, there exists $M > 0$ such that

$$\begin{aligned} f(x^*) - \langle \nabla f(x_n), x^* \rangle + f^*(\nabla f(x_n)) = V_f(x^*, \nabla f(x_n)) \\ = D_f(x^*, x_n) \le M. \end{aligned}$$

Hence, $\{\nabla f(x_n)\}$ is contained in the sublevel set $lev_{\le}^{\psi}(M - f(x^*))$, where $\psi = f^* - \langle ., x^* \rangle$. Since f is lower semicontinuous, f^* is weak* lower semicontinuous. Hence, the function ψ is coercive by Moreau–Rockafellar Theorem (see [38], Theorem 7A and [25]). This shows that $\{\nabla f(x_n)\}$ is bounded. Since f is strongly accretive, f^* is bounded on bounded sets (see [46], Lemma 3.6.1 and [3], Theorem 3.3). Hence ∇f^* is also bounded on bounded subsets of E. (see [13], Proposition 1.1.11). Since f is a Legendre function, it follows that $x_n = \nabla f^*(\nabla f(x_n))$ is bounded for all $n \ge 0$. Therefore $\{x_n\}$ is bounded. So is $\{\nabla f(Tx_n)\}$. Indeed, since f is bounded on bounded subsets of E, ∇f is also bounded on bounded subsets of E (see [13], Proposition 1.1.11). Therefore $\{\nabla f(Tx_n)\}$ is bounded.

Now, following the method of proof in Suantai et al. [40], Zhu and Chang [49] and Mainge [22], we prove the following main theorem.

Theorem 3.2 *Let E be a reflexive real Banach space. Let C be a nonempty, closed and convex subset of E. For each* $j = 1, 2, \ldots, N$, *let* g_j *be a bifunction from* $C \times C$ *satisfying* $(A1) - (A4)$. *Let* $f : E \to \mathbb{R}$ *a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let T be a left Bregman strongly nonexpansive mapping of C into E such that* $F(T) = \widehat{F}(T)$ *and* $\Omega := F(T) \cap (\cap_{j=1}^{N} EP(g_j)) \ne \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* (0,1). *Suppose* $\{x_n\}_{n=1}^{\infty}$ *is iteratively generated by* (15) *with the conditions*

1. $\lim_{n\to\infty} \alpha_n = 0$;
2. $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then, $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $\overleftarrow{Proj}_{\Omega}^f u$, *where* $\overleftarrow{Proj}_{\Omega}^f$ *is the left Bregman projection of E onto* Ω.

Proof Let $z_n := \nabla f^*(\alpha_n \nabla f(u) + \beta_n(1-\alpha_n)\nabla f(x_n) + (1-\alpha_n)(1-\beta_n)\nabla f(Tx_n))$, $n \ge 1$. Furthermore,

$$\begin{aligned} D_f(x^*, x_{n+1}) \le& D_f(x^*, u_{n+1}) \\ =& V_f(x^*, \alpha_n \nabla f(u) + \beta_n(1-\alpha_n)\nabla f(x_n) \\ & + (1-\alpha_n)(1-\beta_n)\nabla f(Tx_n)) \\ \le& V_f(x^*, \alpha_n \nabla f(u) + \beta_n(1-\alpha_n)\nabla f(x_n) \\ & + (1-\alpha_n)(1-\beta_n)\nabla f(Tx_n) - \alpha_n(\nabla f(u) \\ & - \nabla f(x^*)) \\ & - 2\langle \nabla f^*(\alpha_n \nabla f(u) + \beta_n(1-\alpha_n)\nabla f(x_n) \\ & + (1-\alpha_n)(1-\beta_n)\nabla f(Tx_n)) - x^*, \\ & - \alpha_n(\nabla f(u) - \nabla f(x^*))\rangle \\ =& V_f(x^*, \alpha_n \nabla f(x^*) + \beta_n(1-\alpha_n)\nabla f(x_n) \\ & + (1-\alpha_n)(1-\beta_n)\nabla f(Tx_n)) \\ & + 2\alpha_n\langle z_n - x^*, \nabla f(u) - \nabla f(x^*)\rangle \\ =& D_f(x^*, \nabla f^*(\nabla f(x^*) + \beta_n(1-\alpha_n)\nabla f(x_n) \\ & + (1-\alpha_n)(1-\beta_n)\nabla f(Tx_n)) \\ & + 2\alpha_n\langle z_n - x^*, \nabla f(u) - \nabla f(x^*)\rangle \\ \le& \alpha_n D_f(x^*, x^*) + \beta_n(1-\alpha_n) D_f(x^*, x_n) \\ & + (1-\alpha_n)(1-\beta_n) D_f(x^*, Tx_n) \\ & + 2\alpha_n\langle z_n - x^*, \nabla f(u) - \nabla f(x^*)\rangle \\ \le& (1-\alpha_n) D_f(x^*, x_n) + 2\alpha_n\langle z_n - x^*, \nabla f(u) \\ & - \nabla f(x^*)\rangle. \end{aligned} \tag{17}$$

The rest of the proof will be divided into two parts.

Case 1 Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_f(x^*, x_n)\}_{n=n_0}^{\infty}$ is nonincreasing. Then $\{D_f(x^*, x_n)\}_{n=0}^{\infty}$ converges and $D_f(x^*, x_{n+1}) - D_f(x^*, x_n) \to 0, \quad n \to \infty$. Observe that

$$\begin{aligned} D_f(x^*, x_{n+1}) \le D_f(x^*, u_{n+1}) \le \alpha_n D_f(x^*, u) + (1 \\ - \alpha_n) D_f(x^*, x_n). \end{aligned}$$

It then follows that

$$\begin{aligned}D_f(x^*,x_n)-D_f(x^*,Tx_n)=&D_f(x^*,x_n)-D_f(x^*,x_{n+1})\\&+D_f(x^*,x_{n+1})-D_f(x^*,Tx_n)\\\le& D_f(x^*,x_n)-D_f(x^*,x_{n+1})\\&+\alpha_n(D_f(x^*,u)-D_f(x^*,Tx_n))\\&\to 0,\quad n\to\infty.\end{aligned}\tag{18}$$

It then follows that

$$\lim_{n\to\infty} D_f(Tx_n,x_n)=0$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n\_j}\}$ of $\{x_n\}$ that converges weakly to p. Since $F(T)=\widehat{F}(T)$, we have $p\in F(T)$.

Next, we show that $p\in\cap_{k=1}^N EP(g_k)$. Now, using the fact that Res^f_C, $k=1,2,\ldots,N$ is a strictly left quasi-Bregman nonexpansive mapping, we obtain

$$\begin{aligned}D_f(x^*,x_n)=&D_f(x^*,\theta^f_N u_n)\\=&D_f(x^*,Res^f_{g_N}\theta^f_{N-1}u_n)\\\le& D_f(x^*,\theta^f_{N-1}u_n)\le\ldots\le D_f(x^*,u_n).\end{aligned}\tag{19}$$

Since $x^*\in EP(g_N)=F(Res^f_{g_N})$, it follows from Lemma 2.7, (19) and (17) that

$$\begin{aligned}D_f(x_n,\theta^f_{N-1}u_n)=&D_f(Res^f_{g_N}\theta^f_{N-1}u_n,\theta^f_{N-1}u_n)\\\le& D_f(x^*,\theta^f_{N-1}u_n)-D_f(x^*,x_n)\\\le& D_f(x^*,u_n)-D_f(x^*,x_n)\\\le&\alpha_{n-1}M_1+D_f(x^*,x_{n-1})\\&-D_f(x^*,x_n)\to 0,\quad n\to\infty,\end{aligned}$$

for some $M_1>0$. Thus, we obtain $\lim_{n\to\infty} D_f(\theta^f_N u_n,\theta^f_{N-1}u_n)=\lim_{n\to\infty} D_f(x_n,\theta^f_{N-1}u_n)=0$. From Lemma 2.5, we have

$$\lim_{n\to\infty}||\theta^f_N u_n-\theta^f_{N-1}u_n||=\lim_{n\to\infty}||x_n-\theta^f_{N-1}u_n||=0.\tag{20}$$

Since f is uniformly Fréchet differentiable, it follows from Lemma 2.4 and (20) that

$$\lim_{n\to\infty}||\nabla f(\theta^f_N u_n)-\nabla f(\theta^f_{N-1}u_n)||_*=0.\tag{21}$$

Again, since $x^*\in EP(g_{N-1})=F(Res^f_{g_{N-1}})$, it follows from (19) and Lemma 2.7 that

$$\begin{aligned}D_f(\theta^f_{N-1}u_n,\theta^f_{N-2}u_n)=&D_f(Res^f_{g_{N-1}}\theta^f_{N-2}u_n,\theta^f_{N-2}u_n)\\\le& D_f(x^*,\theta^f_{N-2}u_n)-D_f(x^*,\theta^f_{N-1}u_n)\\\le& D_f(x^*,u_n)-D_f(x^*,x_n)\\\le&\alpha_{n-1}M_1+D_f(x^*,x_{n-1})-D_f(x^*,x_n)\\&\to 0,\quad n\to\infty.\end{aligned}$$

Again, we obtain $\lim_{n\to\infty} D_f(\theta^f_{N-1}u_n,\theta^f_{N-2}u_n)=0$. From Lemma 2.5, we have

$$\lim_{n\to\infty}||\theta^f_{N-1}u_n-\theta^f_{N-2}u_n||=0\tag{22}$$

and hence,

$$\lim_{n\to\infty}||\nabla f(\theta^f_{N-1}u_n)-\nabla f(\theta^f_{N-2}u_n)||_*=0.\tag{23}$$

In a similar way, we can verify that

$$\lim_{n\to\infty}||\theta^f_{N-2}u_n-\theta^f_{N-3}u_n||=\cdots=\lim_{n\to\infty}||\theta^f_1 u_n-u_n||=0.\tag{24}$$

From (20), (22) and (24), we can conclude that

$$\lim_{n\to\infty}||\theta^f_k u_n-\theta^f_{k-1}u_n||=0,\quad k=1,2,\ldots,N\tag{25}$$

and

$$\lim_{n\to\infty}||x_n-u_n||=0.$$

Now, since $x_{n_j}\rightharpoonup p$ and $\lim_{n\to\infty}||x_n-u_n||=0$, we obtain that $u_{n_j}\rightharpoonup p$. Again, from (20), (22), (24) and $u_{n_j}\rightharpoonup p$, $n\to\infty$, we have that $\theta^f_k u_n\rightharpoonup p$, $j\to\infty$, for each $k=1,2,\ldots,N$. Also, using (25), we obtain

$$\lim_{n\to\infty}||\nabla f(\theta^f_k u_n)-\nabla f(\theta^f_{k-1}u_n)||_*=0,\quad k=1,2,\ldots,N.\tag{26}$$

By Lemma 2.7, we have that for each $k=1,2,\ldots,N$

$$g_k(\theta^f_k u_{n_j},y)+\langle y-\theta^f_k u_{n_j},\nabla f(\theta^f_k u_{n_j})-\nabla f(\theta^f_{k-1}u_{n_j})\rangle\ge 0,\\ \forall y\in C.$$

Furthermore, using $(A2)$ we obtain

$$\langle y-\theta^f_k u_{n_j},\nabla f(\theta^f_k u_{n_j})-\nabla f(\theta^f_{k-1}u_{n_j})\rangle\ge g_k(y,\theta^f_k u_{n_j}).\tag{27}$$

By $(A4)$, (3.12) and $\theta^f_k u_{n_j}\rightharpoonup p$, we have for each $k=1,2,\ldots,N$

$$g_k(y,p)\le 0,\quad\forall y\in C.$$

For fixed $y\in C$, let $z_{t,y}:=ty+(1-t)p$ for all $t\in(0,1)$. This implies that $z_t\in C$. This yields that $g_k(z_t,p)\le 0$. It follows from (A1) and (A4) that

$$\begin{aligned}0=&g_k(z_t,z_t)\le tg_k(z_t,y)+(1-t)g_k(z_t,p)\\\le& tg_k(z_t,y)\end{aligned}$$

and hence

$$0\le g_k(z_t,y).$$

From condition (A3), we obtain

$$g_k(p,y)\ge 0,\quad\forall y\in C.$$

This implies that $p\in EP(g_k)$, $k=1,2,\ldots,N$. Thus, $p\in\cap_{k=1}^N EP(g_k)$. Hence, we have $p\in\Omega=F(T)\cap(\cap_{k=1}^N EP(g_k))$.

Let $y_n := \nabla f^*\left(\frac{\beta_n(1-\alpha_n)}{1-\alpha_n}\nabla f(x_n) + \frac{(1-\alpha_n)(1-\beta_n)}{1-\alpha_n}\nabla f(Tx_n)\right)$, $n \geq 1$, then

$$D_f(y_n, x_n) \leq \frac{\beta_n(1-\alpha_n)}{1-\alpha_n} D_f(x_n, x_n) + \frac{(1-\alpha_n)(1-\beta_n)}{1-\alpha_n} D_f(Tx_n, x_n) \to 0, \quad n \to \infty. \tag{28}$$

By Lemma 2.5, it follows that $||x_n - y_n|| \to 0, \quad n \to \infty$. Furthermore,

$$\begin{aligned} D_f(y_n, z_n) &= D_f(y_n, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_n)) \\ &\leq \alpha_n D_f(y_n, u) + (1-\alpha_n) D_f(y_n, y_n) \\ &= \alpha_n D_f(y_n, u) \to 0, \quad n \to \infty. \end{aligned} \tag{29}$$

Again, by Lemma 2.5, it follows that $||y_n - z_n|| \to 0, \quad n \to \infty$. Then

$$||x_n - z_n|| \leq ||y_n - z_n|| + ||x_n - y_n|| \to 0, \quad n \to \infty. \tag{30}$$

Let $z := \overleftarrow{Proj}^f_\Omega u$. We next show that $\limsup_{n\to\infty}\langle y_n - z, \nabla f(u) - \nabla f(z)\rangle \leq 0$. To show the inequality $\limsup_{n\to\infty}\langle y_n - z, \nabla f(u) - \nabla f(z)\rangle \leq 0$, we choose a subsequence $\{x_{nj}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n - z, \nabla f(u) - \nabla f(z)\rangle = \lim_{j\to\infty}\langle x_{n_j} - z, \nabla f(u) - \nabla f(z)\rangle.$$

By $||x_n - z_n|| \to 0, \quad n \to \infty$ and Lemma 2.10, we obtain

$$\limsup_{n\to\infty}\langle z_n - z, \nabla f(u) - \nabla f(z)\rangle = \limsup_{n\to\infty}\langle x_n - z, \nabla f(u) - \nabla f(z)\rangle \leq 0. \tag{31}$$

Now, using (31), (17) and Lemma 2.8, we obtain $D_f(z, x_n) \to 0, \quad n \to \infty$. Hence, by Lemma 2.5 we have that $x_n \to z, \quad n \to \infty$.

Case 2 Suppose there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$D_f(x^*, x_{n_i}) < D_f(x^*, x_{n_i+1})$$

for all $i \in \mathbb{N}$. Then by Lemma 2.9, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$D_f(x^*, x_{m_k}) \leq D_f(x^*, x_{m_k+1}) \quad \text{and} \quad D_f(x^*, x_k) \leq D_f(x^*, x_{m_k+1})$$

for all $k \in \mathbb{N}$. Furthermore, we obtain

$$\begin{aligned} D_f(x^*, x_{m_k}) - D_f(x^*, Tx_{m_k}) &= D_f(x^*, x_{m_k}) - D_f(x^*, x_{m_k+1}) \\ &\quad + D_f(x^*, x_{m_k+1}) - D_f(x^*, Tx_{m_k}) \\ &\leq D_f(x^*, x_{m_k}) - D_f(x^*, x_{m_k+1}) \\ &\quad + \alpha_n(D_f(x^*, u) - D_f(x^*, x_{m_k})) \\ &\to 0, \quad k \to \infty. \end{aligned}$$

It then follows that

$$\lim_{k\to\infty} D_f(Tx_{m_k}, x_{m_k}) = 0.$$

By the same arguments as in Case 1, we obtain that

$$\limsup_{k\to\infty}\langle y_{m_k} - z, \nabla f(u) - \nabla f(z)\rangle \leq 0. \tag{32}$$

and

$$D_f(z, x_{m_k+1}) \leq (1-\alpha_{m_k}) D_f(z, x_{m_k}) + 2\alpha_{m_k}\langle \nabla f(u) - \nabla f(z, y_{m_k} - x^*\rangle. \tag{33}$$

Since $D_f(z, x_{m\_k}) \leq D_f(z, x_{m\_k+1})$, we have

$$\alpha_{m_k} D_f(z, x_{m_k}) \leq D_f(z, x_{m_k}) - D_f(z, x_{m_k+1}) + 2\alpha_{m_k}\langle y_{m_k} - z, \nabla f(u) - \nabla f(z)\rangle \leq 2\alpha_{mk}\langle y_{m_k} - z, \nabla f(u) - \nabla f(z)\rangle.$$

In particular, since $\alpha_{mk} > 0$, we get

$$D_f(z, x_{m_k}) \leq 2\langle y_{m_k} - z, \nabla f(u) - \nabla f(z)\rangle. \tag{34}$$

It then follows from (32) that $D_f(z, x_{m_k}) \to 0, \quad k \to \infty$. From (34) and (33), we have

$$D_f(z, x_{m_k+1}) \to 0, \quad k \to \infty.$$

Since $D_f(z, x_k) \leq D_f(z, x_{mk+1})$ for all $k \in \mathbb{N}$, we conclude that $x_k \to z, \quad k \to \infty$. This implies that $x_n \to z, \quad n \to \infty$ which completes the proof.

Corollary 3.3 *Let E be a reflexive real Banach space. Let C be a nonempty, closed and convex subset of E. For each $j = 1, 2, \ldots, N$, let g_j be a bifunction from $C \times C$ satisfying (A1) − (A4).* Let $f : E \to \mathbb{R}$ *a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let T be a left quasi-Bregman firmly nonexpansive mapping of C into E and $\Omega := F(T) \cap (\cap_{j=1}^N EP(g_j)) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in* (0,1). *Suppose $\{x_n\}_{n=1}^\infty$ is iteratively generated by (15) with the conditions*

1. $\lim_{n\to\infty} \alpha_n = 0$;
2. $\sum_{n=1}^\infty \alpha_n = \infty$.

Then, $\{x_n\}_{n=1}^\infty$ converges strongly to $\overleftarrow{Proj}^f_\Omega u$, where $\overleftarrow{Proj}^f_\Omega$ is the left Bregman projection of E onto Ω.

Remark 3.4 Our Theorem 3.2 complements the results of Zhu and Chang [49] in the sense that it can be applied to the approximation of common solution of finite system of equilibrium problems and which is also a fixed point of left Bregman strongly nonexpansive mapping in a reflexive Banach space.

Convergence results concerning family of mappings

In this section, we present strong convergence theorems concerning approximation of common solution to a finite system of equilibrium problems which is also a common fixed point of a family of left Bregman strongly nonexpansive mappings in reflexive real Banach space.

Let C be a subset of a real Banach space E, $f : E \to \mathbb{R}$ a convex and Gâteaux differentiable function and $\{T_n\}_{n=1}^{\infty}$ a sequence of mappings of C such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}_{n=1}^{\infty}$ is said to satisfy the AKTT condition [2] if, for any bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup\{||\nabla f(T_{n+1}z) - \nabla f(T_n z)|| : z \in B\} < \infty.$$

The following proposition is given in the results of Suantai et al. [40].

Proposition 4.1 *Let C be a nonempty, closed and convex subset of a real reflexive Banach space E. Let $f : E \to \mathbb{R}$ be a Legendre and Fréchet differentiable function. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings from C into E such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose that $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT condition. Then there exists the mapping $T : B \to E$ such that*

$$Tx = \lim_{n\to\infty} T_n x, \forall x \in B \tag{35}$$

and $\lim_{n\to\infty} \sup_{z\in B} ||\nabla f(Tz) - \nabla f(T_n z)|| = 0.$

In the sequel, we say that $(\{T_n\}, T)$ satisfies the AKTT condition if $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT condition and T is defined by (35) with $\cap_{n=1}^{\infty} F(T_n) = F(T)$.

By following the method of proof of Theorem 3.2, method of proof Theorem 4.2 of Suantai et al. [40] and Proposition 4.1, we prove the following theorem.

Theorem 4.2 *Let E be a reflexive real Banach space. Let C be a nonempty, closed and convex subset of E. For each $j = 1, 2, ..., N$, let G_j be a bifunction from $C \times C$ satisfying $(A1) - (A4)$. Let $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of left Bregman strongly nonexpansive mappings on C such that $F(T_n) = \widehat{F}(T_n)$ for all $n \geq 0$ and $\Omega := (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{j=1}^{N} EP(g_j)) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in (0,1). Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by by u, $u_0 \in E$,*

$$\begin{cases} x_n = Res^f_{g_N} Res^f_{g_{N-1}} \ldots Res^f_{g_2} Res^f_{g_1} u_n, \\ u_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)(\beta_n \nabla f(x_n) \cdot \\ \qquad + (1-\beta_n)\nabla f(T_n x_n))), \quad n \geq 1, \end{cases} \tag{36}$$

with the conditions

1. $\lim_{n\to\infty} \alpha_n = 0$;
2. $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If $(\{T_n\}, T)$ satisfies the AKTT condition, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $\overleftarrow{Proj}^f_{\Omega} u$, where $\overleftarrow{Proj}^f_{\Omega}$ is the left Bregman projection of E onto Ω.

Next, using the idea in [32], we consider the mapping $T : C \to C$ defined by $T = T_m \ T_{m-1} ... T_1$, where $T_i (i = 1, 2, \ldots, m)$ are left Bregman strongly nonexpansive mappings on E. Using Theorem 3.2 and Theorem 4.3 of Suantai et al. [40], we proof the following theorem.

Theorem 4.3 *Let E be a reflexive real Banach space. Let C be a nonempty, closed and convex subset of E. For each $j = 1, 2, ..., N$, let g_j be a bifunction from $C \times C$ satisfying $(A1) - (A4)$. Let $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $T_i (i = 1, 2, \ldots, m)$ be a sequence of left Bregman strongly nonexpansive mappings on C such that $F(T_i) = \widehat{F}(T_i)$ for all $n \geq 0$ and $\Omega := (\cap_{i=1}^{m} F(T_i)) \cap (\cap_{j=1}^{N} EP(g_j)) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in (0,1). Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by by u, $u_0 \in E$,*

$$\begin{cases} x_n = Res^f_{g_N} Res^f_{g_{N-1}} \ldots Res^f_{g_2} Res^f_{g_1} u_n, \\ u_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)(\beta_n \nabla f(x_n) \quad . \\ \qquad + (1-\beta_n)\nabla f(T_m T_{m-1} ... T_1 x_n))), \quad n \geq 1, \end{cases} \tag{37}$$

with the conditions

1. $\lim_{n\to\infty} \alpha_n = 0$;
2. $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}_{n=1}^{\infty}$converges strongly to $\overleftarrow{Proj}^f_{\Omega} u$,where $\overleftarrow{Proj}^f_{\Omega}$is the left Bregman projection of E onto Ω.

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